A Glimpse Into Operator Theory: *Part I*

Robert F. Allen

George Mason University

Graduate Seminar
February 7, 2006

Definition

glimpse *n.*

1. *A brief, incomplete view or look.*
2. *Archaic.* *A brief flash of light.*

Outline of Part I

Banach and Hilbert Spaces

Definition

A **normed linear space** X *is a vector space (over* $\mathbb{C}$*) with norm* $||\cdot|| : X \to [0, \infty)$ *such that*

1. $||x|| = 0$ *iff* $x = 0$,
2. $||x + y|| \leq ||x|| + ||y||$,
3. $||\alpha x|| = |\alpha| \, ||x||$.

For a normed linear space X, we define a metric

$$d(x, y) = ||x - y|| .$$

Examples

1. $\mathbb{C}^n$:

$$||(z_1, z_2, \ldots, z_n)|| = \left(\sum_{j=1}^{n} |z_j|^2 \right)^{1/2}$$

2. $C(\mathcal{Y}) = \{f : \mathcal{Y} \to \mathbb{C} \mid f \text{ cont. and } \mathcal{Y} \text{ compact Hausdorff}\}$:

$$||f|| = \max_{y \in \mathcal{Y}} \{|f(y)|\}$$

3. $L^p(X, \mu)$ *for positive measure* μ:

$$||f||_p = \begin{cases} \left(\int_X |f|^p \, d\mu \right)^{1/p}, & (1 \leq p < \infty) \\ \sup_{x \in X} \{|f(x)|\}, & (p = \infty) \end{cases}$$

Definition

Let X be a normed linear space. If X is complete in the induced metric, then X is called a **Banach space***.*

Bloch Space

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

$$\mathcal{B} = \left\{ f \text{ analytic on } \mathbb{D} : \underbrace{\sup_{z \in \mathbb{D}} (1 - |z|^2) \left| f'(z) \right|}_{\beta_f} < \infty \right\}.$$

$\mathcal{B}$ is a Banach space under the norm

$$||f||_{\mathcal{B}} = |f(0)| + \beta_f .$$

Definition

An **inner product space** X *is a vector space over* $\mathbb{C}$ *with a map* $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{C}$ *such that*

1. $\langle x, y \rangle = \overline{\langle y, x \rangle}$,
2. $\langle x, x \rangle \geq 0$ *with* $\langle x, x \rangle = 0$ *iff* $x = 0$,
3. $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$,
4. $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$.

For an inner product space X, we define a norm

$$||x|| = \langle x, x \rangle^{1/2} .$$

Examples

1. $\mathbb{C}^n$:

$$\langle (z_1, \ldots, z_n), (w_1, \ldots, w_n) \rangle = \sum_{i=1}^{n} z_i \overline{w_i}$$

2. ℓ^2:

$$\langle (z_1, z_2, \ldots), (w_1, w_2, \ldots) \rangle = \sum_{i=1}^{\infty} z_i \overline{w_i}$$

3. $L^2(X, \mu)$ *for positive measure* μ:

$$\langle f, g \rangle = \int_X f \overline{g} \, d\mu$$

Definition

Let X be an inner product space. If X is complete in the induced metric, then X is called a (complex) **Hilbert space***.*

Bergman space

$$A^2(\mathbb{D}) = L_a^2(\mathbb{D}) = \left\{ f \text{ analytic on } \mathbb{D} : \int_{\mathbb{D}} |f(z)|^2 \, \frac{dA}{\pi} < \infty \right\}.$$

$L_a^2(\mathbb{D})$ is a closed subspace of $L^2\left(\mathbb{D}, \frac{dA}{\pi}\right)$.

Definition

For $f, g \in \mathcal{H}$, we say $f \perp g$ if $\langle f, g\rangle = 0$. We define

$$A^{\perp} = \{h \in \mathcal{H} : \langle h, a\rangle = 0 \ \forall a \in \mathcal{A}\}.$$

Theorem (Projection Theorem)

Let M be a closed subspace of $\mathcal{H}$. Then there exists a unique pair $P : \mathcal{H} \to M$ and $Q : \mathcal{H} \to M^{\perp}$ such that $x = Px + Qx, \forall x \in \mathcal{H}$. Furthermore

1. $\forall x \in M$, $Px = x$ *and* $Qx = 0$,
2. $\forall x \in M^{\perp}$, $Px = 0$ *and* $Qx = x$,
3. P *and* Q *are linear maps,*
4. $||x||^2 = ||Px||^2 + ||Qx||^2$.

P and Q are called the **orthogonal projections**.

Bounded Linear Operators

Definition

A linear map $T : X \to Y$ between normed linear spaces is called a **bounded linear operator** *if $\exists M > 0$ such that, for all $x \in X$*

$$||Tx|| \leq M\,||x||\,.$$

We denote the set of such bounded linear operators by $\mathscr{B}(X, Y)$.

Proposition

If $T : X \to Y$ is a linear map between normed linear spaces, TFAE:

1. *T is continuous,*
2. *T is continuous at $x = 0$,*
3. *T is bounded.*

Proof.

Suppose T is continuous at $x = 0$. For $\varepsilon = 1$, there exists $\delta > 0$ such that $||Tx|| \leq 1$ whenever $||x|| \leq \delta$. For $v \neq 0$,

$$||Tv|| = \left|\left| \frac{||v||}{\delta} T\left(\delta \frac{v}{||v||}\right) \right|\right| = \frac{||v||}{\delta} \left|\left| T\left(\delta \frac{v}{||v||}\right) \right|\right| \leq \frac{1}{\delta} ||v|| .$$

Suppose T is bounded. So there exists M such that $||Tv|| \leq M ||v||$ for all $v \in \mathcal{H}$. Let $\varepsilon > 0$. Define $\delta = \frac{\varepsilon}{M}$ and suppose $||x - y|| < \delta$. We see that

$$||Tx - Ty|| = ||T(x - y)|| \leq M ||x - y|| < M\delta = \varepsilon.$$

$\square$

Definition

If $T : X \to Y$ is a bounded linear operator, then

$$\begin{aligned} ||T|| &= \sup\{||Tx|| : ||x|| \leq 1\}, \\ &= \sup\{||Tx|| : ||x|| = 1\}, \\ &= \sup\left\{\frac{||Tx||}{||x||} : ||x|| \neq 0\right\}, \\ &= \inf\{C : ||Tx|| \leq C\,||x||\}. \end{aligned}$$

$\mathscr{B}(X, Y)$ is a normed linear space under $||\cdot||$, and if Y is complete then $\mathscr{B}(X, Y)$ is a Banach space.

$\mathscr{B}(X, \mathbb{C})$ is a Banach space and $\Lambda \in \mathscr{B}(X, \mathbb{C})$ is called a **bounded linear functional**.

Orthogonal Projections

$P : \mathcal{H} \to M$ is a linear operator on Hilbert space $\mathcal{H}$ into a closed subspace M.

$$||Ph||^2 \leq ||h||^2 .$$

So $||P|| \leq 1$. If $h \in M$, then $||Ph|| = ||h||$. So $||P|| = 1$.

Multiplication Operators

$M_\varphi(f) = \varphi f$ with symbol $\varphi \in L^\infty(X, \mu)$ is a linear operator from $L^2(X, \mu)$ to $L^2(X, \mu)$ for positive measure space (X, μ).

$$||M_\varphi(f)||^2 = \int_X |\varphi f|^2 \, d\mu \leq ||\varphi||_\infty^2 \, ||f||_2^2 .$$

So $||M_\varphi|| \leq ||\varphi||_\infty$.

Shift Operators on ℓ^2

The forward shift operator $S : \ell^2 \to \ell^2$ is defined as $S(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$.

$$||S(x_1, x_2, \dots)|| = ||(0, x_1, x_2, \dots)|| = ||(x_1, x_2, \dots)|| .$$

So $||S|| = 1$.

The backward shift operator $S^* : \ell^2 \to \ell^2$ is defined as $S^*(x_1, x_2, \dots) = (x_2, x_3, \dots)$.

$$||S^*(x_1, x_2, \dots)|| = ||(x_2, x_3, \dots)|| \leq ||(x_1, x_2, \dots)|| .$$

So $||S^*|| \leq 1$. If $x = (0, x_2, x_3, \dots)$, then $||S^*x|| = ||x||$. Thus $||S^*|| = 1$.

Toeplitz Operators

$T_\varphi : L^2_a(\mathbb{D}) \to L^2_a(\mathbb{D})$ is a linear operator with symbol $\varphi \in L^\infty\left(\mathbb{D}, \frac{dA}{\pi}\right)$. We define $T_\varphi(f) = PM_\varphi(f)$.

From the previous two operators, $||T_\varphi|| \leq ||\varphi||$.

Composition Operators

A composition operator is of the form $C_\varphi(f) = f \circ \varphi$. The first question is to determine what symbol φ makes C_φ a bounded linear operator.

On $L^2_a(\mathbb{D})$, define $C_\varphi : L^2_a(\mathbb{D}) \to L^2_a(\mathbb{D})$ with symbol φ an analytic self-map of $\mathbb{D}$ by $C_\varphi(f) = f \circ \varphi$.

For this space, $||C_\varphi|| = 1$. This is not true for most spaces, and is not true for $L^2_a(\mathbb{B}_N)$, $N > 1$.

Adjoints of Bounded Linear Operators

Definition

Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces. A **sesqui-linear form** $u : \mathcal{H} \times \mathcal{K} \to \mathbb{C}$ *is a mapping satisfying*

1. $u(\alpha h + \beta g, k) = \alpha u(h, k) + \beta u(g, k)$,
2. $u(h, \alpha k + \beta \ell) = \overline{\alpha} u(h, k) + \overline{\beta} u(h, \ell)$.

We say u is **bounded** *if $\exists M > 0$ such that $|u(h, k)| \leq M \, ||h|| \, ||k||$ for all $h \in \mathcal{H}$ and $k \in \mathcal{K}$.*

Theorem

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and $u : \mathcal{H} \times \mathcal{K} \to \mathbb{C}$ be a bounded sesqui-linear form. Then there exists unique $A \in \mathscr{B}(\mathcal{H}, \mathcal{K})$ and $B \in \mathscr{B}(\mathcal{K}, \mathcal{H})$ such that $u(h, k) = \langle Ah, k \rangle = \langle h, Bk \rangle$.

Definition

For $A \in \mathscr{B}(\mathcal{H}, \mathcal{K})$, the **adjoint of** *$A$, denoted A^* is the unique element of $\mathscr{B}(\mathcal{K}, \mathcal{H})$ such that $\langle Ah, k \rangle = \langle h, A^* k \rangle$. A is called* **self-adjoint** *if $A = A^*$,* **normal** *if $A^* A = AA^*$, and* **unitary** *if $A^* A = AA^* = I$.*

Properties of Adjoints

For $A, B \in \mathscr{B}(\mathcal{H})$:

1. $A^{**} = A$,
2. $(A + B)^* = A^* + B^*$,
3. $(\alpha A)^* = \overline{\alpha} A^*$,
4. $(AB)^* = B^* A^*$,
5. $||A|| = ||A^*|| = ||AA^*||^{1/2}$.

Orthogonal Projections

Let $P : \mathcal{H} \to \mathcal{M}$ be the orthogonal projection into a closed subspace. Then, for all $f, g \in \mathcal{H}$,

$$\langle Ph, g \rangle = \langle h, Pg \rangle .$$

Let $f = m_1 + n_1$ and $g = m_2 + n_2$ where $m_1, m_2 \in M$ and $n_1, n_2 \in M^\perp$. $Pg = m_2$ and $Pf = m_1$ by uniqueness.
$\langle f, Pg \rangle = \langle m_1 + n_1, m_2 \rangle = \langle m_1, m_2 \rangle = \langle m_1, m_2 + n_2 \rangle = \langle Pf, g \rangle$.

So $P = P^*$.

Multiplication Operators

For $M_\varphi : L^2(X,\mu) \to L^2(X,\mu)$, we have

$$\langle M_\varphi f, g\rangle = \int_X \varphi f \overline{g}\, d\mu = \int_X f \overline{\overline{\varphi} g}\, d\mu = \langle f, \overline{\varphi} g\rangle = \langle f, M_{\overline{\varphi}} g\rangle .$$

So $M_\varphi^* = M_{\overline{\varphi}}$ and M_φ is self-adjoint iff φ is real.

Shift Operators on ℓ^2

For $x = (x_1, x_2, \dots), y = (y_1, y_2, \dots) \in \ell^2$

$$\begin{aligned}\langle Sx, y\rangle = \langle (0, x_1, x_2, \dots), (y_1, y_2, y_3, \dots)\rangle &= \sum_{i=1}^{\infty} x_i \overline{y_{i+1}} \\ &= \langle (x_1, x_2, \dots), (y_2, y_3, \dots)\rangle = \langle x, S^* y\rangle .\end{aligned}$$

So $(S)^* = S^*$.

Toeplitz Operators

For T_φ acting on $L^2_a(\mathbb{D})$,

$$\begin{aligned}\langle T_\varphi f, g\rangle &= \langle PM_\varphi f, g\rangle = \langle M_\varphi f, Pg\rangle_{L^2} = \langle M_\varphi f, g\rangle_{L^2} \\ &= \langle f, M_{\overline{\varphi}} g\rangle_{L^2} = \langle Pf, M_{\overline{\varphi}} g\rangle_{L^2} = \langle f, PM_{\overline{\varphi}} g\rangle_{L^2} \\ &= \langle f, PM_{\overline{\varphi}} g\rangle = \langle f, T_{\overline{\varphi}} g\rangle .\end{aligned}$$

So $T_\varphi^* = T_{\overline{\varphi}}$.

Composition Operators

For C_φ acting on $L^2_a(\mathbb{D})$, we can write down an expression for C_φ^* for φ a linear fractional map;

$$\varphi(z) = \frac{az+b}{cz+d}.$$

$C_\varphi^* = M_g C_\sigma T_{\overline{h}}$ where

$$\sigma(z) = \frac{\overline{a}z - \overline{c}}{-\overline{b}z + \overline{d}}$$
$$g(z) = \frac{1}{(-\overline{b}z + \overline{d})^2}$$
$$h(z) = (cz+d)^2.$$

Preview of Part II

1. Operator Algebras
2. Spectral Theory
3. Maximal Ideal Spaces

References

- Conway, J., *A Course in Functional Analysis*, Springer, 1997.
- Cowen, C. & Maccluer, B., *Composition Operators on Spaces of Analytic Functions*, CRC-Press, 1995.
- Duran, P. & Schuster, A., *Bergman Spaces*, AMS Press, 2004.
- Rudin, W., *Real & Complex Analysis*, McGraw-Hill, 1987.